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THE STIFFNESSES OF ELASTIC CYLINDRICAL BEAMS†

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Continuing the analysis in [1, 2] of the problem of the theory of elasticity in regions of small diameter, methods derived from the asymptotic theory for calculating the stiffnesses of cylindrical beams are considered, and the results are compared with those of classical theory [3, 4]. The technique of two-scale expansions [5–8], as formulated in [1], is employed (in this case the averaging methods for homogeneous problems are not applicable [9]. It is shown that, if Poisson's ratio ν is constant, the stiffnesses of a beam may be computed from formulae derived from the classical theory of plane sections, though the local deformations need not generally coincide. If $\nu \neq$ const the stiffnesses differ from their classical values. Two-sided estimates are obtained for that case. The classical stiffnesses are exact lower bounds.

The paper by Kozlova [10] is of some relevance to the questions considered here.

1. STATEMENT OF THE PROBLEM

Consider an elastic body occupying a cylindrical region $[-1,\ 1]\times\varepsilon S = Q_\varepsilon$ of characteristic diameter $\varepsilon \ll 1$. The beam cross-section S is a connected region with smooth boundary. The local elastic constants a_{ijkl} are functions of x_2/ε, x_3/ε. It has been shown [1] that as $\varepsilon \to 0$ the solution of the problem of the theory of elasticity in Q_ε tends to the solution of the problem of the theory of beams with defining relations

$$\begin{aligned} N_{11} &= \varepsilon^2 A_1^0 e_{11} - \varepsilon^3 A_{1\alpha}^1 \rho_\alpha + \varepsilon^3 B_{11}^0 \varphi' \\ M_{\beta 1} &= \varepsilon^3\, {}^1A_{1\beta} e_{11} - \varepsilon^4 A_{\alpha\beta 1}^2 \rho_\alpha + \varepsilon^4 B_{\beta 1}^1 \varphi' \\ M &= \varepsilon^3 A e_{11} - \varepsilon^4 A_\alpha^2 \rho_\alpha + \varepsilon^4 B^1 \varphi' \end{aligned} \tag{1.1}$$

where e_{11} is the axial deformation ($e_{11} = V_1'$, where V_1 is the axial displacement), ρ_α are the curvatures ($\rho_\alpha = u_\alpha^{(0)''}$ $u_\alpha^{(0)}$ being the deflections, $\alpha = 2,\ 3$), φ is the angle of twist (φ' is the torsion), N_{11} is the axial force, $M_{\alpha 1}$ are the bending moments, M is the torque, and the prime denotes differentiation with respect to x_1.

We will consider the cellular problems (CP) of the theory of beams [1], which is

$$\begin{aligned} &(a_{\alpha\beta\gamma\delta} X_{\gamma,\delta}^{AB} + (-1)^{A-1} a_{\alpha\beta 11} y_B^{A-1})_{,\beta} = 0 \ \text{ in } S \\ &(a_{\alpha\beta\gamma\delta} X_{\gamma,\delta}^{AB} + (-1)^{A-1} a_{\alpha\beta 11} y_B^{A-1}) n_\beta = 0 \ \text{on } \partial S \end{aligned} \tag{1.2}$$

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The arguments in (1.2) are y_2, y_3 ($\mathbf{y}=\mathbf{x}/\varepsilon$ are dimensionless variables), $\beta=\partial/\partial y_\beta$; $\alpha, \beta, \gamma, \delta = 2, 3$; $A, B = 1, 2$; $\mathbf{n}=(n_2, n_3)$ is the normal to ∂S.

The beam stiffnesses are computed using the following formulae [1]

$$A_1^0 = \langle a_{1111} + a_{11\gamma\delta} X^{11}_{\gamma,\delta}\rangle \tag{1.3}$$

$$A^1_{1\alpha} = \langle -a_{1111} y_\alpha + a_{11\gamma\delta} X^{2\alpha}_{\gamma,\delta}\rangle \tag{1.4}$$

$$^1A_{1\beta} = \langle y_\beta (a_{1111} + a_{11\gamma\delta} X^{11}_{\gamma,\delta})\rangle \tag{1.5}$$

$$A^2_{\alpha\beta i} = -\langle y_\beta (a_{i111} y_\alpha + a_{i1\gamma\delta} X^{2\alpha}_{\gamma,\delta})\rangle \tag{1.6}$$

$$B^0_{11} = \langle a_{111\alpha} X^3_{1,\alpha}\rangle, \quad B^1_{\alpha\beta} = \langle y_\beta a_{\alpha 1\gamma\delta} X^3_{\gamma,\delta}\rangle \tag{1.7}$$

$$^1A = {}^1A_{23} - {}^1A_{32}, \quad A^2_\alpha = A^2_{\alpha 23} - A^2_{\alpha 32}, \quad B = B^1_{32} - B^1_{32} \tag{1.8}$$

$$\langle \cdot \rangle = \int_S dy_2 dy_3$$

We have here used X^3_1 to denote the solution of the CP representing twisting of the beam [1]

$$\begin{aligned} &(a_{1\delta 1\gamma} X^3_{1,\gamma} + a_{1\delta\beta 1} s_\beta y_{\tilde\beta})_{,\delta} = 0 \quad \text{in } S \\ &(a_{1\delta 1\gamma} X^3_{1,\gamma} + a_{1\delta\beta 1} s_\beta y_{\tilde\beta}) n_\delta = 0 \quad \text{on } \partial S \end{aligned} \tag{1.9}$$

where $\tilde\beta = 3$ for $\beta = 2$ and $\tilde\beta = 2$ for $\beta = 3$; $s_1 = 0$, $s_3 = -1$.

The displacements of the beam, treated as a three-dimensional body, have the following form [1]

$$\mathbf{u}^{(0)}(x_1) + \varepsilon \mathbf{u}^{(1)}(x_1, \mathbf{y}) + \varepsilon^2 \mathbf{u}^{(2)}(x_1, \mathbf{y}) + \ldots \tag{1.10}$$

$$(\mathbf{y} = \mathbf{x}/\varepsilon, \quad \mathbf{V} = \mathbf{V}(x_1), \quad \varphi = \varphi(x_1), \quad u^{(0)}_\alpha = u^{(0)}_\alpha(x_1), \quad u^{(0)}_1 = 0 \tag{1.11}$$

$$u^{(1)}_1 = -y_\alpha u^{(0)\prime}_\alpha + V_1, \quad u^{(1)}_\beta = S_\beta y_{\tilde\beta} \varphi + V_\beta \tag{1.12}$$

$$\begin{aligned} u^{(2)}_1 &= X^{11}_1(\mathbf{y}) V_1' - y_\alpha V_\alpha' + X^{2\alpha}_1(\mathbf{y}) u^{(0)\prime\prime}_\alpha + X^3_1(\mathbf{y})\varphi' \\ u^{(2)}_\beta &= X^{11}_\beta(\mathbf{y}) V_1' + X^{2\alpha}_\beta(\mathbf{y}) u^{(0)\prime\prime}_\alpha + X^3_\beta(\mathbf{y})\varphi' \\ &\alpha, \beta = 2, 3 \end{aligned} \tag{1.13}$$

Formulae (1.11)–(1.13) define the local deformations of the beam as a three-dimensional body. The terms of order ε in (1.10) correspond to the hypothesis of plane sections [3], but the stiffnesses are also determined by terms involving the solution of the CP, i.e. by terms of the order of ε^2.

Henceforth we shall assume that the beam is made from an isotropic material:

$$a_{ijkl} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.14}$$

(δ_{ij} is the Kronecker delta).

It will be seen that the classification in this situation is based on the condition: $\nu = \text{const}$ or $\nu \neq \text{const}$.

2. THE CASE ν = const

2.1. *Tensile stiffness of the beam*

The solution of the CP (1.2) with $A=1$ is $X_\beta^{11}=-\nu y_\beta$. Substituting it into (1.3) we obtain

$$A_1^0=\langle E\rangle \tag{2.1}$$

which is precisely the formula derived on the basis of the plane section hypothesis.

2.2. *Flexural stiffness of the beam*

The solution of problem (1.2) with $A=2$ is (the problem is solved for $B=2$, corresponding to bending in the Ox_2x_3 plane)

$$X_2^{22}=\nu\frac{y_2^2}{2}-\nu\frac{y_3^2}{2},\quad X_3^{22}=\nu y_2 y_3 \tag{2.2}$$

As can be verified, if X_2^{22}, X_3^{22} are given by (2.2), then

$$a_{\alpha\beta\gamma\delta}X_{\gamma,\delta}^{22}-a_{\alpha\beta 11}=0$$

for any E. The solution of the CP with $B=3$ is obtained from (2.2) by interchanging the indices $2\rightleftarrows 3$.

Substituting (2.2) into (1.6), we obtain the classical formula

$$A_{\alpha\beta 1}^2=\langle Ey_\beta y_\alpha\rangle \tag{2.3}$$

Here, however, the local deformations of the beam in the cross-section are not those predicted by the plane section hypothesis.

We note that, as formula (2.3) holds for an arbitrary beam cross-section S, elongation of the region S along one of its axes does not give formulae for the flexural stiffness of a plate. This indicates that the expansions in [1] for beams and in [11] for plates, despite their similarity, are essentially different.

3. THE CASE $\nu\neq$ const

Here it will be shown that the classical formulae are not generally true in the asymptotic theory. The conclusions of the asymptotic theory agree with those of the three-dimensional theory of elasticity [4]. We shall consider an example in which E is constant, ν is almost constant and the classical method of small parameters may be used [12].

We define the stress function F^A [13] by

$$s_{22}=F_{,33}^A,\quad s_{33}=F_{,22}^A,\quad s_{23}=s_{32}=-F_{,23}^A \tag{3.1}$$

$$s_{\alpha\beta}=a_{\alpha\beta\gamma\delta}X_{\gamma,\delta}^{AB}+(-1)^{A-1}a_{\alpha\beta 11}y_B^{A-1} \tag{3.2}$$

In this case the equilibrium equations of the CP (1.2) are satisfied identically, while the boundary conditions of (1.2) give

$$F^A=\partial F^A/\partial\mathbf{n}=0\quad\text{on}\quad\partial S \tag{3.3}$$

Using Hooke's law (1.14), we deduce from (3.2) that

$$e_{22}=\frac{1-\nu^2}{E}F^A_{,33}-\frac{\nu(1-\nu^2)}{E}F^A_{,22}+\nu y^{A-1}_B$$
$$e_{33}=-\frac{\nu(1-\nu^2)}{E}F^A_{,33}+\frac{1-\nu^2}{E}F^A_{,22}+\nu y^{A-1}_B \tag{3.4}$$
$$e_{23}=e_{32}=-\frac{1+\nu}{E}F^A_{,23}$$

where $e_{\alpha\beta}=(X^{AB}_{\alpha,\beta}+X^{AB}_{\beta,\alpha})/2$ are the deformations, which must satisfy the compatibility condition [13]

$$e_{22,33}+e_{33,22}-2e_{23,23}=0$$

In view of (3.4), this condition may be written ($E=\text{const}$, Δ denotes the Laplace operator)

$$L(\nu)F^A\equiv[(1-\nu^2)F^A_{,33}-\nu(1-\nu^2)F^A_{,22}]_{,33}+[(1-\nu^2)F^A_{,22}-\nu(1-\nu^2)F^A_{,33}]_{,22}+$$
$$+2[(1+\nu)F^A_{,23}]_{,23}+E^{-1}\Delta(\nu y^{A-1}_B)=0 \text{ в } S \tag{3.5}$$

Suppose Poisson's ratio has the form $\nu=\nu_0+\delta\nu^{(1)}$, $\nu_0=\text{const}$, $\delta\ll1$, $\|\nu^{(1)}\|_{C^{(1)}(S)}\leq1$.
If $\delta=0$, problem (3.5), (3.3) has a trivial solution. The perturbed solution has the form

$$F^A=0+\delta F^{(1)A}+\delta^2F^{(2)A}+\dots \tag{3.6}$$

Substituting (3.6) into (3.5), (3.3) and equating the coefficients of δ, we obtain the following problem for $F^{(1)A}$

$$L(\nu_0)F^{(1)A}=-\frac{1}{E}\Delta(\nu^{(1)}y^{A-1}_B) \text{ in } S$$
$$F^{(1)A}=\partial F^{(1)A}/\partial\mathbf{n}=0 \text{ on } \partial S \tag{3.7}$$

It follows from the stiffnesses in (1.3), (1.6) and (1.14) and from formulae (3.4) that

$$A^0_1=\langle\xi-\eta\Delta F^{(1)}\rangle,\quad A^2_{\alpha\beta1}=\langle y_\beta(\xi y_\alpha-\eta\Delta F^{(2)})\rangle$$
$$\xi=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)},\quad \eta=\frac{E\nu(1-\nu)^2}{1-2\nu} \tag{3.8}$$

Using the formula

$$\frac{\nu(1-\nu)^2}{1-2\nu}=\frac{\nu_0(1-\nu_0)^2}{1-2\nu_0}+\left[1+\frac{3\nu_0^2-4\nu_0^3}{(1-2\nu_0)^2}\right]\delta\nu^{(1)}+\dots=a+b\delta\nu^{(1)}+\dots$$

we obtain expressions for the perturbed values of the quantities (3.8)

$$E\,\text{mes}\,S-Ea\delta\langle\Delta F^{(1)1}\rangle-\delta b^2\langle\nu^{(1)}\Delta F^{(1)1}\rangle-\delta^2a\langle\Delta F^{(1)2}\rangle+\dots$$
$$E\langle y_\alpha y_\beta\rangle-Ea\delta\langle\Delta F^{(1)2}y_\beta\rangle-\delta^2b\langle\nu^{(1)}\Delta F^{(1)2}y_\beta\rangle-\delta^2a\langle\Delta F^{(2)2}y_\beta\rangle+\dots \tag{3.9}$$

We have retained here terms of order δ^2 (see (3.6)). Note that by (3.3) and (3.6), $F^{(i)A}=\partial F^{(i)A}/\partial\mathbf{n}=0$ on ∂S, and integration by parts yields

$$\langle\Delta F^{(i)A}y^{A-1}_\beta\rangle=\int_S F^{(i)A}\Delta y^{A-1}_\beta dy_2dy_3+$$

$$+\int_{\partial S}\frac{\partial F^{(i)A}}{\partial \mathbf{n}}y_\beta^{A-1}dy_2dy_3-\int_{\partial S}F^{(i)A}\frac{\partial y_\beta^{A-1}}{\partial \mathbf{n}}dy_2dy_3=0$$

Thus, the perturbation of the stiffness, by which they differ from those predicted by the classical formulae, are

$$\delta^2 b\langle \nu^{(1)}\Delta F^{(1)A}y_\beta^{A-1}\rangle \tag{3.10}$$

Multiplying (3.7) by $F^{(1)C}$ and integrating by parts, using the boundary conditions for $F^{(1)A}$, and putting $\nu_0=0$, we obtain from (3.10)

$$\delta^2\langle \Delta F^{(1)\alpha}\Delta F^{(1)\beta}\rangle,\quad (\alpha,\beta)=(0,0),\ (1,1)$$

This expression may be non-zero when $\alpha=\beta$.

4. TWISTING

When $E=\text{const}$ and $\nu=\text{const}$, the problem is the same as in classical theory, as may be verified using the CP (1.9) and formula (1.7).

5. ESTIMATES FOR THE STIFFNESSES OF NON-HOMOGENEOUS CYLINDRICAL BEAMS

5.1. *Flexural stiffnesses*

As will be evident in what follows, the functionals for computing the beam stiffnesses are most naturally evaluated by starting from the three-dimensional CP. The CP for flexural stiffnesses (see [1]) is

$$\begin{aligned}&(a_{ijkl}X^\alpha_{k,l}-a_{ij11}y_\alpha)_{,j}=0 \ \text{ in } P\\&(a_{ijkl}X^\alpha_{k,l}-a_{ij11}y_\alpha)n_j=0 \ \text{ on } \ \gamma\\&,j=\partial/\partial y_j;\ \ i,j,k,l=1,2,3;\ \ \alpha=2,3\end{aligned}\tag{5.1}$$

The function $\mathbf{X}^\alpha(\mathbf{y})$ is periodic in y_1 with period 1, and $P=[0,\ 1]\times S$ is a three-dimensional element of the beam (see Fig. 1).

The flexural stiffnesses are given by

$$A^2_{\alpha\beta 1}=\langle y_\beta(a_{1111}y_\alpha-a_{11kl}X^\alpha_{k,l})\rangle \tag{5.2}$$

Remark. Problem (3.1) has the following solution [1]

$$X^\alpha_1=0,\quad X^\alpha_\beta=X^\alpha_\beta(y_2,y_3) \tag{5.3}$$

Substitution of this solution into (5.1) and (5.2) gives problem (1.2) and formula (1.6)

Conversion of (5.2) *into quadratic forms*. The reduction of the initial problem to extremum problems is fundamental for deriving two-sided estimates [14, 15]. We will derive formulae that yield the flexural stiffnesses $A^2_{\alpha\beta 1}$ as extremum values of certain functionals.

Multiply Eq. (5.1) by X^β_i, add and integrate by parts over P taking into account the boundary conditions of (5.1). This gives

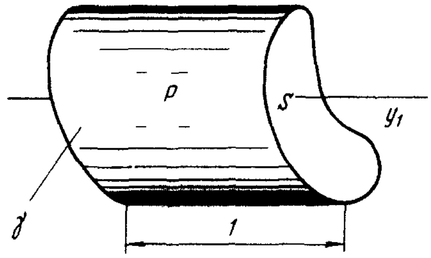

Fig. 1.

$$\langle (a_{ijkl}X^{\alpha}_{k,l} - a_{ijkl}y_{\alpha}\delta_{k1}\delta_{l1})X^{\beta}_{i,j}\rangle_P = 0$$
$$\langle\cdot\rangle_P = \int_P dy_1 dy_2 dy_3 \tag{5.4}$$

Formula (5.2) becomes

$$A^2_{\alpha\beta 1} = \langle y_{\beta}(a_{ijkl}\delta_{k1}\delta_{l1}y_{\alpha} - a_{ijkl}X^{\alpha}_{k,l})\delta_{i1}\delta_{j1}\rangle_P \tag{5.5}$$

Subtracting (5.5) from (5.4), we get

$$A^2_{\alpha\beta 1} = \langle a_{ijkl}(X^{\alpha}_{k,l} - \delta_{k1}\delta_{l1}y_{\alpha})(X^{\beta}_{i,j} - \delta_{i1}\delta_{j1}y_{\beta})\rangle_P \tag{5.6}$$

The relationship between (5.6) *and the Lagrange and Castigliano functionals of problem* (5.1). Let us consider the Lagrange functional $J_u(\mathbf{X})$ and the Castigliano functional $J_{\sigma}(\sigma)$ for the CP (5.1) [11, 13]

$$J_u(\mathbf{X}) = -\langle a_{ij11}y_{\alpha}X_{ij}\rangle_P - \frac{1}{2}\langle a_{ijkl}X_{i,j}X_{k,l}\rangle_P$$
$$J_{\sigma}(\sigma) = \frac{1}{2}\langle a^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}\rangle_P \tag{5.7}$$

where the virtual displacements [13] $\mathbf{X}$ are periodic in y_1 with period 1 and the admissible stresses [13] $\{\sigma_{ij}\}$ belong to the set $\Sigma = \{\sigma_{ij} : (\sigma_{ij} - a_{ij11}y_{\alpha})_{,j} = 0$ in P, $(\sigma_{ij} - a_{ij11}y_{\alpha})n_j = 0$ on γ, σ_{i1} are periodic in y_1 with period 1$\}$. Here $\{a^{-1}_{ijkl}\}$ is the tensor inverse to $\{a_{ijkl}\}$.

It is well known that [13, 14]

$$\max_{\mathbf{X}} J_u(\mathbf{X}) = \min_{\sigma_{ij}\in\Sigma} J_{\sigma}(\sigma) \tag{5.8}$$

and problem (5.1) is Euler's equation for $J_u(\mathbf{X})$. Hence it follows from (5.6)–(5.8), after transforming for $\alpha = \beta$, that

$$\max_{\mathbf{X}} J_u(\mathbf{X}) = -\frac{1}{2}(A^2_{\alpha\alpha 1} - \langle a_{1111}y^2_{\alpha}\rangle_P) = \min_{\sigma_{ij}\in\Sigma} J_{\sigma}(\sigma)$$

and after some algebra we have the two-sided estimate

$$\langle a_{ijkl}(X_{k,l} - \delta_{k1}\delta_{l1}y_{\alpha})(X_{i,j} - \delta_{i1}\delta_{j1}y_{\alpha})\rangle_P \geqslant A^2_{\alpha\alpha 1} \geqslant \langle a_{1111}y^2_{\alpha}\rangle_P - \langle a^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}\rangle_P \tag{5.9}$$

for any function $\mathbf{X}$ periodic in y_1 with period 1 and any function $\{\sigma_{ij}\} \in \Sigma$.

Estimate (5.9) *for cylindrical beams*. For the case in question the set of admissible displacements $\mathbf{X}$ reduces to (5.3), and the corresponding set of stresses is described by the following conditions (it is here that the general three-dimensional CP is used)

$$\begin{aligned}(\sigma_{\alpha\beta} - a_{\alpha\beta 11} y_A)_{,\beta} &= 0 \text{ in } S \\ (\sigma_{\alpha\beta} - a_{\alpha\beta 11} y_A) n_\beta &= 0 \text{ on } \partial S\end{aligned} \tag{5.10}$$

where $\sigma_{11} = \sigma_{11}(y_2, y_3)$ is any sufficiently smooth function, $\sigma_{1\alpha} = 0$, $\alpha = 2, 3$.

Thus, for cylindrical beams the minimization/maximization region may be reduced to (5.3) and (5.10). examining inequality (5.9) in the sets (5.3) and (5.10) we obtain

$$\langle a_{1111} y_A^2 \rangle_P + \langle a_{\alpha\beta\gamma\delta} X_{\alpha,\beta} X_{\gamma,\delta} \rangle_P \geqslant A_{AA1}^2 \tag{5.11}$$

$$A_{AA1}^2 \geqslant \langle a_{1111} y_A^2 \rangle_P - \langle a_{\alpha\beta\gamma\delta}^{-1} \sigma_{\alpha\beta} \sigma_{\gamma\delta} \rangle_P - 2\langle a_{\alpha\beta 11}^{-1} \sigma_{\alpha\beta} \sigma_{11} \rangle_P - \langle a_{1111} \sigma_{11}^2 \rangle_P \tag{5.12}$$

on the assumption that $\mathbf{X}$ and $\{\sigma_{ij}\}$ satisfy conditions (5.3) and (5.10), respectively.

Noting that σ_{11} in (5.10) is an arbitrary function, one can independently maximize the right-hand side of (5.12) as a function of σ_{11}. To that end it will suffice to solve the problem

$$2\langle a_{\alpha\beta 11}^{-1} \sigma_{\alpha\beta} \sigma_{11} \rangle_P + \langle a_{1111} \sigma_{11}^2 \rangle_P \to \min$$

By (5.3), Euler's equation for this problem is (since $\langle\,\rangle_P = \langle\,\rangle$)

$$a_{\alpha\beta 11}^{-1} \sigma_{\alpha\beta} + a_{1111}^{-1} \sigma_{11} = 0$$

As a result, $\sigma_{11} = -a_{\alpha\beta 11}^{-1} \sigma_{\alpha\beta} / a_{1111}^{-1}$, and substituting this expression into (5.12) we obtain the estimate

$$A_{AA1}^2 \geqslant \langle a_{1111} y_A^2 \rangle - \left\langle a_{\alpha\beta\gamma\delta}^{-1} \sigma_{\alpha\beta} \sigma_{\gamma\delta} \right\rangle + \left\langle \frac{(a_{\alpha\beta 11}^{-1} \sigma_{\alpha\beta})^2}{a_{1111}^{-1}} \right\rangle \tag{5.13}$$

where $\sigma_{\alpha\beta}$ satisfy (5.10).

An estimate of the stiffnesses for cylindrical domains. Estimates (5.11) and (5.13) are exact for the set of admissible functions (5.3) and (5.10) (that is, the minimum of the left-hand side of (5.11) and the maximum of the right-hand side of (5.13) are the same). If the test functions are selected at random, one obtains two-sided estimates. For $\mathbf{X}$, take $\mathbf{X} = 0$. It follows from (5.11) that

$$A_{\alpha\alpha 1}^2 \leqslant \langle a_{1111} y_\alpha^2 \rangle = \left\langle \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} y_\alpha^2 \right\rangle \tag{5.14}$$

Take the admissible field of displacements to be

$$\sigma_{22} = \sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)} \nu y_\alpha, \quad \sigma_{23} = \sigma_{32} = 0 \tag{5.15}$$

Substituting (5.15) into (5.13), after some algebra (noting that

$$a_{\alpha\alpha\gamma\gamma}^{-1} \sigma_{\gamma\gamma} = \frac{1}{E} \delta_{\alpha\alpha} - \frac{\nu}{E} \sigma_{\gamma\gamma}$$

for $\gamma \neq \alpha$), we obtain

$$-\langle a^{-1}_{\alpha\beta\gamma\delta}\sigma_{\alpha\beta}\sigma_{\gamma\delta}\rangle + \left\langle \frac{(a^{-1}_{\alpha\beta 11}\sigma_{\alpha\beta})^2}{a^{-1}_{1111}} \right\rangle = -\left\langle \frac{2\nu^2 E y_\alpha^2}{(1+\nu)(1-2\nu)} \right\rangle$$

Hence we deduce, using (1.14), that the right-hand side of (5.13) equals $\langle Ey_\alpha^2\rangle$. Combining the last estimate with (5.14), we obtain the following two-sided estimate

$$\left\langle \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} y_\alpha^2 \right\rangle \geqslant A^2_{\alpha\alpha 1} \geqslant \langle E y_\alpha^2 \rangle \tag{5.16}$$

As follows from Section 2 above, equality is achieved on the right-hand side of (5.16), e.g. when $\nu = \text{const}$. By Section 3, one can also achieve strict inequality on the right-hand side of (5.16). If $\nu \neq \text{const}$, we have a representation of the form $A^2_{\alpha\alpha 1}(\nu) = A^2_{\alpha\alpha 1}(\nu_0) + A(\nu - \nu_0, \ \nu - \nu_0)$. The absence of a linear term in the expansion of the averaged characteristics is typical for averaging procedures [16, 17].

the range (i.e. the difference between the upper and lower bounds) in (5.16) is $\langle 2\nu^2 E(1+\nu)^{-1}(1-2\nu)^{-1} y_\alpha^2\rangle$. For materials encountered in practice, $0.2 \leqslant \nu \leqslant 0.4$ and the range is bounded by the number $\langle Ey_\alpha^2\rangle$.

5.2. *Tensile stiffnesses*

The CP for elongation of the beam is [1]

$$\begin{aligned} (a_{ijkl}X^1_{k,l} - a_{ij11})_{,j} &= 0 \quad \text{in } P \\ (a_{ijkl}X^1_{k,l} - a_{ij11})n_j &= 0 \quad \text{on } \gamma \end{aligned} \tag{5.17}$$

The function $\mathbf{X}^1(\mathbf{y})$ is periodic in y_1 with period 1.

The tensile stiffness is computed by the formula

$$A^0_1 = \langle a_{1111} + a_{11kl}X^1_{k,l}\rangle_P$$

One can proceed as before for this formula and the CP (5.17) (see sub-section 5.1), yielding the following two-sided estimate for the tensile stiffness of the beam

$$\left\langle \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \right\rangle \geqslant A^0_1 \geqslant \langle E \rangle$$

5.3. *Torsional stiffness*

The CP and the formula for torsional stiffness in the asymptotic theory are similar to those of classical theory. As two-sided estimates for the classical cases have already been developed, we need not discuss them here.

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